Notes
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Based on

1 Introduction

2 Conditional Probability

Tower Property: For X and Y random variables

$$E[E(X|Y)] = E(X)$$

Or

$$E[E(X|Y)] = \begin{cases} \sum_{y \in S} E(X|Y=y) p_Y(y) & \text{if } Y \text{ is discrete} \\ \int_{-\infty}^{\infty} E(X|Y=y) f_Y(y) dy & \text{if } Y \text{ is continuous} \end{cases}$$

3 Stochastic Processes and Discrete Time Markov Chains

Stochastic Process A stochastic process $(X_t)_{t \in T}$ is an indexed collection of random variables. Set T is the index set, in general interpreted as time, and X_t is the state of the system at time t. The set S of all possible states is referred as the state space of the process.
Definition: Given that we start at i, the probability of ever reaching state j (after finite steps) is

$$\varrho_{ij} = P(T_j < \infty | X_0 = i) = \sum_{n=1}^{\infty} P(T_j = n | X_0 = i)$$

Recurrent: if $\varrho_{ii} = 1$ || iff $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$
Transient: if $\varrho_{ii} < 1$
Definition: m_{ii} is the mean time to return to state i

$$m_{ij} = E(T_j | X_0 = i)$$

Definition: The mean proportion of time spent in state j when starting from i.

$$p_{ij}^* = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{n} p_{ij}^{(n)}$$

Null Recurrent: when $m_{ii} = \infty$
Positive Recurrent: when $m_{ii} < \infty$ || recurrent state is positive recurrent iff $p_{ij}^* = \frac{1}{m_{ii}} > 0$.
If $i \leftrightarrow j$ then either i and j are both positive recurrent or both null recurrent.

Exponential Random Variable

Cumulative Distribution Function:

$$P(X \leq x) = F(x) = 1 - e^{-\lambda x}$$

Density:

$$f(x) = \lambda e^{-\lambda x}$$

$$E(X^r) = \frac{r!}{\lambda^r}$$

Memoryless Property: Exponential distribution has a memoryless propety, where for real numbers s, t

$$P(X > s + t | X > s) = \frac{X > s + t, X > s}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t)$$

Strong Memoryless Property: if X_2 is an exponential random variable with rate λ and X_1 is an independent non-negative continuous random variable, then for any $x \geq 0$

$$P(X_2 > X_1 + x | X_2 > X_1) = P(X_2 > x) = e^{-\lambda x}$$

Sums of Exponentials: if $Z = X_1 + X_2 + \ldots + X_n$, where $X_i \sim \exp(\lambda)$ for all i and independent, then Z is called the gamma (n, λ) random variable and its density function is

$$f_n(z) = \lambda e^{-\lambda z} \frac{(\lambda z)^{n-1}}{(n-1)!}$$

Poisson Processes

Poisson Process: Let τ_i be an independent exponential (λ) random variables, $S_0 = 0, S_n = \tau_1 + \ldots + \tau_n$ and $N_t = \max\{n \geq 0 : S_n \leq t\}$. Then $(N_t)_{t \in \mathbb{R}_{\geq 0}}$ is a Poisson Process with rate parameter λ, or PP(λ). With distribution of rate λt

$$P(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

Future Events: A PP(λ) counts the events starting from time 0. Suppose we reset the counter at time s and start counting the future events, define this as

$$N_t^{(s)} = N_{s+t} - N_s \quad \text{for all } t \geq 0$$

Where $(N_t^{(s)})_{t \geq 0}$ is a PP(λ) and it is dependent of $(N_u)_{0 \leq u \leq s}$

Stationary Increments: A process $(N_t)_{t \geq 0}$ is said to have stationary increments if $N_{s+t} - N_s$ is identically distributed for all s, i.e. the distribution does not depend on s.
Independent Increments: A process $(N_t)_{t \geq 0}$ has independent increments if $N_{s_1+t_1} - N_{s_1}$ and $N_{s_2+t_2} - N_{s_2}$ are independent if $(s_1, s_1 + t_1] \cap (s_2, s_2 + t_2] = \emptyset$.

So $(N_t)_{t \geq 0}$ is a PP(λ) iff:

1. It has stationary and independent increments
2. N_t is Poisson (λt) random variable for all t.

Little o: A function $f(x)$ is $o(x)$ if $\lim_{x \to 0} \frac{f(x)}{x} = 0$.
So $(N_t)_{t \geq 0}$ is a PP(λ) iff:

1. It has stationary and independent increments
2. $$\begin{aligned} &P(N_h = 0) = 1 - \lambda h + o(h) \\ &P(N_h = 1) = \lambda h + o(h) \\ &P(N_h \geq 2) = o(h) \end{aligned}$$

Superpositioning and Splitting

Campbell's Theorem

Looknig at a single arrival, ask if it occured before time s, with $s \leq t$:

$$P(S_1 \leq s | N_t = 1) = \frac{s}{t}$$

In general, the probability of k events happened before time s with the condition that n events happened till time t

$$P(N_s = k | N_t = n) = \begin{pmatrix} n \\ k \end{pmatrix} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$$

Campbell's Theorem: Let S_n be the even times for a PP. If $N_t = n$ is given then the vector $(S_1, S_2, \ldots, S_n)$ follows the distribution of ordered independent uniform variables $(U_{(1)}, U_{(2)}, \ldots, U_{(n)})$. Consequently, the unordered set of arrival times $\{S_1, \ldots, S_n\}$ has the same distribution as $\{U_1, \ldots, U_n\}$.

Nonhomogeneous Poisson Process

Counting Process: A counting process $(N_t)_{t\geq 0}$ is a nonhomogeneous PP if

1. It has independent increments
2. $$\begin{aligned} P\left(N_{t+h} - N_t = 0\right) &= 1 - \lambda(t)h + o(h) \\ P\left(N_{t+h} - N_t = 1\right) &= \lambda(t)h + o(h) \\ P\left(N_{t+h} - N_t \geq 2\right) &= o(h) \end{aligned}$$

Nonhomogeneous Poisson Process: Define

$$\Lambda(t) = \int_0^t \lambda(u) du$$

Then N_t is a Poisson $\Lambda(t)$ random variable for any t.
$N_t - N_s$ is a Poisson random variable with parameter $\Lambda(t) - \Lambda(s) = \int_s^t \lambda(u)du$.
The superpositioning and splitting properties are true for nonhomogeneous Poisson processes as well.

Event Times for Nonhomogeneous PP: Gives a binomial with n trials and a success probability of $\Lambda(s)/\Lambda(t)$.

$$P(N_s = k | N_t = n) = \begin{pmatrix} n \\ k \end{pmatrix} \left(\frac{\Lambda(s)}{\Lambda(t)}\right)^k \left(1 - \frac{\Lambda(s)}{\Lambda(t)}\right)^{n-k}$$

Compound Poisson Processes

Suppose events occur according to a non-homogeneous PP$(\lambda(t))$. When event n occurs we incur a random cost of Y_n, which is independent and identically distributed for all n. Then the total cost incurred up to time t is given by

$$Z_t = \sum_{n=1}^{N_t} Y_n$$

and $(Z_t)_{t\geq 0}$ is a Compound Poisson Process with

$$E(Z_t) = \Lambda(t)E(Y_1)$$

$$\text{Var}(Z_t) = \Lambda(t)E(Y_1^2)$$

Continuous Time Markov Chains

CTMC: The continuous time stochastic process $(X_t)_{t\geq 0}$ is called a CTMC if:

1. Each duration τ_n is an exponential random variable with rate $q_i > 0$ which depends only on state $\tilde{X}_{n-1} = i$ the process leaves
2. The corresponding (embedded) discrete time process $(\tilde{X}_n)_{n\in\mathbb{N}}$ is a DTMC with $\tilde{p}_{ii} = 0$ for all i.

Markov: A continuous time process $(X_t)_{t\geq 0}$ has the Markov property if for any $0 \leq s_0 < s_1 < \ldots < s_n < s$, any $t \geq 0$ and any possible states $i_0, \ldots, i_n, i, j$ we have:

$$P(X_{s+t} = j | X_s = i, X_{s_n} = i_n, \ldots, X_{s_0} = i_0) = P(X_{s+t} = j | X_s = i)$$

Rate Matrix: or generator Q of the CTMV $(X_t)_{t\geq 0}$ is defined through its elements

$$q_{ij} = q_i \tilde{p}_{ij}, \text{ if } i \neq j$$

$$q_{ii} = -q_i = -\sum_{j\in S, j\neq i} q_{ij}$$

Here q_{ij} is the jump rate from i to j.
Transition Matrix:

$$p_{ij}(t) = P(X_t = j | X_0 = i)$$

Initial Distribution:

$$a_i^{(0)} = P(X_0 = i)$$

$P(t)$ has the following properties

1. $p_{ij}(t) \geq 0$
2. $\sum_{j\in S} p_{ij}(t) = 1$ for all $t \geq 0$
3. (Chapman-Kolmogorov Equations) $P(t+s) = P(t)P(s)$, that is $p_{ij}(t+s) = \sum_{k\in S} p_{ik}(t) p_{kj}(s)$

Theorem: $P'(0) = Q$, that is

$$p'_{ii}(0) = -q_i = q_{ii} \quad , \quad p'_{ij}(0) = q_{ij} \quad \text{for } i \neq j$$

Backward and Forward Equations

Let $P(t)$ be the transition matrix and Q be the generator of a CTMC. Then $P(t)$ is the unique solution of both the forward and backward Kolmogrov equation with initial conditions

$$P(0) = \hat{1} \quad \text{that is} \quad p_{ii}(0) = 1, p_{ij}(0) = 0 \text{ for } j \neq i$$

Forward Kolmogrov Equation:

$$P'(t) = P(t)Q \quad \text{that is } p'_{ij}(t) = \sum_{k\in S} p_{ik}(t) q_{kj}$$

Backward Kolmogrov Equation:

$$P'(t) = QP(t) \quad \text{that is } p'_{ij}(t) = \sum_{k\in S} q_{ik} p_{kj}(t)$$

Theorem: For finite state spaces the solution of both backward and forward equations is $P(t) = e^{Qt}$.

Transience and Recurrence

Define

$$\rho_{ij} = P\left(T_j < \infty | X_0 = i\right), \quad m_{ij} = E\left(T_j | X_0 = i\right)$$

Then we have: State i is

1. Transient if $\rho_{ii} < 1$
2. Null Recurrent if $\rho_{ii} = 1$ and $m_{ii} = \infty$
3. Positive Recurrent if $\rho_{ii} = 1$ and $m_{ii} < \infty$

Theorem: A state i for a CTMC is recurrent iff the embedded DTMC is recurrent.
Theorem: Let $(X_t)_{t\geq 0}$ be an irreducable CTMC. Suppose $\tilde{\pi}$ is a positive solution $\tilde{\pi} = \tilde{\pi}\tilde{P}$ where $\tilde{P}$ is the transition matrix of the embedded DTMC. Then the CTMC is positive recurrent iff

$$\sum_{i\in S} \frac{\tilde{\pi}_i}{q_i} < \infty$$

Stationary Probabilities

Limiting Distribution: Let $(X_t)_{t\geq 0}$ be an irreducible CTMV with limiting distribution π_i. The limiting distribution is the unique stationary distribution, that is the unique solution of the global balance equations

$$\pi Q = 0, \quad \sum_{i\in S} \pi_i = 1$$

iff the CTMC is positive recurrent.

Note: the stationary distribution π of a CTMC is different than the stationary distribution $\tilde{\pi}$ of the embedded DTMC.
Theorem: If the detailed balance equations are satisfied, that is

$$\pi_i q_{ij} = \pi_j q_{ji} \quad \text{for all } i \neq j, \text{ and } \sum_i \pi_i = 1$$

then also the global balance equations are satisfied, so π is the stationary distribution.

First Passage Properties

Cost and Rewards

Definitions

Examples

Density function: $f(x)$
Distribution of X: $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$
Expected value: $E(X) = \int_{-\infty}^{\infty} xf(x)dx$